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SUPERRIGIDITY, RATNER'S THEOREM, AND FUNDAMENTAL GROUPS

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ABSTRACT

We discuss the implications of superrigidity and Ratner's theorem on invariant measures on homogeneous spaces for understanding the fundamental group of manifolds with an action of a semisimple Lie group.

If a non-compact simple Lie group acts on a manifold M, possibly preserving some geometric structure, it is natural to enquire as to the possible relationship between the structure of G and that of $\pi = \pi_1(M)$. Results in this direction appear in the work of Gromov [1], the author [7], and Spatzier and the author [3].

A basic tool in some of these results is the superrigidity theorem for cocycles [4], or, in more invariant terms, the superrigidity theorem for actions of G on principal bundles. In this paper we show how to obtain further results on this question by combining techniques of superrigidity with Ratner's recent solution to the Raghunathan conjecture on invariant measures on homogeneous spaces. This appears to be a new direction of application of Ratner's fundamental work. An interesting feature of our results is that the results in [3],[7],[9] give "lower bounds" on the possible fundamental groups given the presence of a G-action, while in this work we are able to give a type of "upper bound".

Let G be a connected simple Lie group with finite center and with $\mathbf{R}\text{-rank}(G) \geq 2$. We assume G acts continuously on a compact manifold M preserving a probabil-

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ity measure. Then $G^{\sim}$ acts on $M^{\sim}$. We recall briefly the notions of engaging and topologically engaging from [7]. Namely, the action is called engaging if there is no loss of ergodicity in passing to finite covers of M, and topologically engaging if some element $g \in G^{\sim}$ both projects to an element not contained in a compact subgroup of G and acts tamely on $M^{\sim}$, e.g., acts properly on a conull set. A fundamental result of Gromov is that if the G action is real analytic and preserves a real analytic connection, then the action of $G^{\sim}$ on the universal cover of M is proper on a conull set, and in particular is topologically engaging. All known examples of smooth G actions preserving a volume are both engaging and topologically engaging. The action of $G^{\sim}$ on the principal bundle $M^{\sim} \to M$ yields a measurable cocycle $\alpha \cdot G^{\sim} \times M \to \pi$, with the property that for each $g \in G^{\sim}$, $\alpha(g,\cdot)^{\pm 1}$ is a bounded function on M. Therefore, for any representation $\sigma : \pi \to GL(n,\mathbf{C})$, we clearly have $\sigma(\alpha(g,\cdot)^{\pm 1})$ is bounded. This enables us to freely apply the results of [8], which yields information on the structure of such cocycles under these (in fact weaker) boundedness conditions.

We now recall the statements of superrigidity and of Ratner's theorem in the form in which we shall need them.

THEOREM 1. (Superrigidity [4],[8]) *Let G and M be as above, and let H be a product of finitely many (rational points of) connected algebraic groups defined over (possibly varying) local fields of characteristic* 0. *Let $\lambda : G^{\sim} \times M \to H$ be a cocycle with the property that $\lambda(g,\cdot)^{\pm 1}$ is bounded for each g. Assume the action of G on M is ergodic.*

(a) *Assume the algebraic hull of the cocycle* [4] *is (algebraically) connected. Then there is a continuous homomorphism $\theta : G^{\sim} \to H$ and a compact subgroup $K \subset H$ commuting with $\theta(G^{\sim})$ such that λ is equivalent to a cocycle of the form $\beta(g,m) = \theta(g)c(g,m)$ where $c(g,m) \in K$.*

(b) *With no connectivity assumption on the algebraic hull, we can obtain the same conclusion by lifting λ to a cocycle on a finite extension of M.*

THEOREM 2. (Ratner [2]) *Let H be a connected Lie group, $\Gamma \subset H$ a discrete subgroup, and $G \subset H$ a connected simple non-compact subgroup. Let ν be a finite G-invariant ergodic measure on H/Γ, where the action is given by the embedding of G in H. Then there is a closed connected subgroup L with $G \subset L \subset H$ and a point $x \in H/\Gamma$ say with stabilizer in H being $h\Gamma h^{-1}$ such that*:

(i) *$L \cap h\Gamma h^{-1}$ is a lattice in L; and*

(ii) *ν is the measure on H/Γ corresponding to the invariant volume on $L/L \cap h\Gamma h^{-1}$ under the natural bijection $L/L \cap h\Gamma h^{-1} \cong Lx \subset H/\Gamma$.*

The next lemma is the observation that one can combine these results.

LEMMA 3. *Let G and H be as in Theorem* 1, *and write $H = H_\infty \times H_f$ where H_∞ is the product of the real and complex terms, and H_f the product of the totally disconnected terms. Let $\Gamma \subset H$ be a discrete subgroup, and assume that $\Gamma \cap H_f = \{e\}$. Let M be an ergodic G-space with a finite invariant measure, and let $\lambda : G^{\sim} \times M \to \Gamma \subset H$ be a cocycle with the property that $\lambda(g, \cdot)^{\pm 1}$ is bounded for each g. We also assume that this cocycle is not equivalent (as a cocycle into Γ) to a cocycle into a finite subgroup of Γ. Then, either with the assumption that the algebraic hull of λ is Zariski connected, or alternatively, by passing to a finite ergodic extension of M, we have:*

There is a non-trivial homomorphism $\theta : G^{\sim} \to H_\infty$ (which automatically factors to a homomorphism of a finite cover of G), and a closed subgroup L, $\theta(G^{\sim}) \subset L \subset H_\infty$, and a compact subgroup $C \subset L$ commuting with $\theta(G^{\sim})$ such that:

(i) *Γ contains a subgroup isomorphic to a lattice $\Gamma' \subset L$; in fact we can take Γ' to be the intersection of L with a conjugate of $p_\infty(\Gamma)$, where p_∞ is the projection of H onto H_∞.*

(ii) *There is a measure preserving $G^{\sim}$-map $M \to C\backslash L/\Gamma'$, where the measure on the latter derives from the projection of the L-invariant volume form on L/Γ'; in particular $\theta(\pi_1(G))$ acts trivially on $C\backslash L/\Gamma'$, and hence we have a G-map $M \to C\backslash L/\Gamma'$.*

For the proof we will need the following easy fact.

LEMMA 4. *Let $\alpha : M \times G \to \Gamma$ be a cocycle for any ergodic group action taking values in a countable group Γ. Suppose $i : \Gamma \to L$ is an embedding of Γ as a discrete subgroup of a locally compact group L. If $i \circ \alpha$ is equivalent to a cocycle into a compact subgroup $K \subset L$, the α is equivalent to a cocycle into a finite subgroup of Γ.*

PROOF OF LEMMA 3. Choose θ, β, c, and K as in Theorem 1. By Lemma 4, θ is non-trivial. Thus, there is a measurable map $\psi : M \to H$ such that $\psi(gm)\alpha(g,m)\psi(m)^{-1} = c(m,g)\theta(g)$. Rewriting this as $c(m,g)^{-1}\psi(gm)\alpha(g,m) = \theta(g)\psi(m)$, we see that in $K\backslash H/\Gamma$ we have $\omega(gm) = \theta(g)\omega(m)$ where ω is the composition of ψ with the projection $H \to K\backslash H/\Gamma$. If μ is the G-invariant measure on M, let $\omega_*(\mu)$ be the projection of this measure to $K\backslash H/\Gamma$. Thus, we have a $\theta(G^{\sim})$-invariant, ergodic, probability measure on $K\backslash H/\Gamma$. Let ν_1 be the lift of this measure to H/Γ by taking the image of Haar measure on each of the fibers of $H/\Gamma \to K\backslash H/\Gamma$. This measure is a $\theta(G^{\sim})$-invariant probability measure but it is not *a priori* ergodic. However, the arguments of [5] show that any ergodic component, say ν, still projects onto $\omega_*(\mu)$. Fix such a ν, which is now a $\theta(G^{\sim})$-invariant, ergodic, probability measure on $(H_\infty \times H_f)/\Gamma$.

Let $T \subset H_f$ be a compact open subgroup. By enlarging K if necessary, we may assume T to be chosen such that $T \subset K$. Let $\Gamma_\infty = \Gamma \cap (H_\infty \times T)$. Since T is compact, the projection of Γ_∞ to H_∞ (which we recall is injective on Γ by assumption) is still discrete. We can identify $(H_\infty \times T)/\Gamma_\infty \subset (H_\infty \times H_f)/\Gamma$ as an open $G^{\sim}$-invariant subset. Since T is open in H_f, we can find a countable number of translates of $(H_\infty \times T)/\Gamma_\infty$ by elements of H_f whose union covers all of $(H_\infty \times H_f)/\Gamma$. Since H_f commutes with $\theta(G^{\sim})$ these translates are all open and $G^{\sim}$ invariant, and it follows that ν is supported on one of these translates, say translation by $h \in H_f$. We may thus translate ν itself back via h^{-1} to $(H_\infty \times T)/\Gamma_\infty$, and obtain a $G^{\sim}$-invariant ergodic measure on this set, and by projection, a $G^{\sim}$-invariant ergodic measure, say ν_2 on H_∞/Γ_∞, where we have identified Γ_∞ with its image under projection. Assertion (i) now follows immediately from Ratner's theorem. To see (ii), we simply observe that we may view ω as a $G^{\sim}$-map $M \to K\backslash(H \times hT)/\Gamma_\infty$, and via projection we obtain an equivariant map $M \to p(K)\backslash H_\infty/\Gamma_\infty$, where p is the projection. Since the image of μ under this map is clearly the same as the image of ν_2, (ii) follows.

As discussed in [7], for an action which is engaging or topologically engaging, the $\pi_1(M)$ valued cocycle, say α, defined by the lift of the action to the universal cover of M is not equivalent to a cocycle into a finite subgroup.

COROLLARY 5. *Let G be as in Theorem* 1, *and suppose G acts on a compact manifold M, preserving a finite measure. Suppose the action is either engaging, topologically engaging, or real analytic connection and volume preserving. If $\pi_1(M)$ is isomorphic to a discrete subgroup of a Lie group, it contains a lattice in a Lie group L which contains a subgroup locally isomorphic to G.*

In particular, this applies if we take $\pi_1(M)$ to be a subgroup of $\mathrm{GL}(n,\mathcal{O})$ for some n, where $\mathcal{O}$ is the ring of algebraic integers.

COROLLARY 6. *Let G and M be as in Corollary* 5. *Suppose $\pi_1(M)$ is isomorphic to a lattice Γ in G. Then the action is measurably isomorphic to an action induced from an action of a lattice commensurable with* Γ.

PROOF. Since Γ is Zariski dense in G, we must clearly have $L = G$ and C being trivial in Lemma 3.

From Corollary 6, we also recover the main results of [9].

COROLLARY 7. *Let G and M be as in Corollary 5. Suppose there is a faithful representation* $\pi_1(M) \to \mathrm{GL}(n,\mathbf{Q}^{\sim})$ *for some n. Then* $\pi_1(M)$ *contains a lattice in a Lie group L where L contains a group locally isomorphic to G.*

This follows from the fact that any finitely generated subgroup of $\mathrm{GL}(n,\mathbf{Q}^{\sim})$ is isomorphic to a discrete subgroup in a product of algebraic groups as in Lemma 3. From Corollary 7 we can also recover a number of the results of [7] concerning faithful representations.

One obtains a rather different type of result by combining Lemma 3 with some considerations of entropy. Namely, from Lemma 3 one can sometimes deduce that simply by knowing the fundamental group of M, any (engaging or topologically engaging) G-action on M must have entropy bounded below by specific algebraic data. This in turn can place dimension restrictions on M given the fundamental group. To describe this precisely, we introduce some notation.

DEFINITION 8. Let $\Gamma \subset L$ be a lattice in a connected Lie group L, and let G be another Lie group. If $H \subset L$ is a closed connected subgroup and $\Lambda \subset H$ is a lattice in H, we call the pair (H,Λ) G-related (or (G,θ)-related if more precision is required) if there is some non-trivial homomorphism $\theta: G^{\sim} \to L$ with $\theta(G^{\sim}) \subset H \subset L$, and some conjugate $\Gamma' = \lambda\Gamma\lambda^{-1}$ of Γ in L with $H \cap \Gamma' = \Lambda$. We shall also call a subgroup $\Gamma_1 \subset \Gamma$ G-related if there is a G-related (H,Λ) with $\Gamma_1 = \lambda^{-1}\Lambda\lambda$, with λ as above.

We can rephrase Lemma 3 in this context as follows. We recall that a homomorphism of groups is called an isogeny if it is surjective with finite kernel.

COROLLARY 9. *Let G and M be as in Corollary 5. Assume G acts ergodically on M. Suppose there is an isogeny* $\pi_1(M) \to \Gamma$*, where* Γ *is a lattice in a connected Lie group L. Then there is a finite ergodic extension M′ of M, a G-related pair* (H,Λ) *in L, a compact subgroup* $C \subset H$ *commuting with the image of* θ*, and a measurable G-equivariant measure preserving map* $M' \to C\backslash H/\Lambda$.

If G acts on a space X with invariant probability measure, we let $h(g,X)$ denote the Komogorov–Sinai entropy of the transformation defined by g. Since entropy is unchanged by a finite ergodic extension, or more generally by an isometric extension (i.e., via a homogeneous space of a compact group [5]), it follows that with the notation of Corollary 9 we have for each $g \in G$, $h(g,M) \geq h(g,H/\Lambda)$. We recall that the entropy on H/Λ can be computed from purely algebraic information.

Namely, if A is a matrix, we define its entropy $h(A)$ to be $\sum \log(|\lambda|)$, where λ runs through the eigenvalues of A of magnitude at least 1. Then for any $a \in H$, $h(a, H/\Lambda) = h(\mathrm{Ad}_H a)$. Further, given any G-invariant probability measure on M, its entropy can be computed by integrating the entropy over the ergodic components. Therefore, we have:

Corollary 10. *Let G and M be as in Corollary 5. Suppose there is an isogeny $\pi_1(M) \to \Gamma$, where Γ is a lattice in a connected Lie group L. Then with respect to any G-invariant probability measure on M, we have for any $g \in G^{\sim}$ that*

$$h(g, M) \geq \min\{h(\mathrm{Ad}_H(\theta(g))) \mid \textit{where } H \subset L \textit{ is } (G,\theta)\textit{-related}\}.$$

Now consider the case in which M is a smooth compact manifold with a G-invariant volume. The entropy of each $g \in G$ can the be computed via Pesin's formula, which in light of Theorem 1 yields a close relationship between $\dim(M)$ and possible values of the entropy. See [4] for a discussion. We can formulate this as follows. For each positive integer m, and $g \in G^{\sim}$, let

$$c_m(g) = \max\{h(\pi(g)) \mid \pi : G^{\sim} \to \mathrm{GL}(m, \mathbf{R}) \text{ is a linear representation}\}.$$

This of course is completely calculable in principle, and is certainly easily calculated in low dimensions. Then by [4] we have for any smooth volume preserving action of G on M with $\dim(M) \leq m$ that $h(g, M) \leq c_m(g)$. We therefore deduce:

Corollary 11. *Let G be as in Theorem 1. Let M be a compact smooth manifold of dimension m, with a volume preserving action of G. Assume the action is either engaging or topologically engaging (e.g., C^ω connection preserving). Suppose there is an isogeny $\pi_1(M) \to \Gamma$, where Γ is a lattice in a connected Lie group L. Then*

$$c_m(g) \geq \min\{h(\mathrm{Ad}_H(\theta(g))) \mid \textit{where } H \subset L \textit{ is } (G,\theta)\textit{-related}\}.$$

This result can be viewed as giving an upper bound on certain features of Γ if there is an action of G on a compact manifold of dimension at most m and with fundamental group Γ. Corollary 11 is of little use in the extreme case in which there is a representation θ such that $\theta(G)$ intersects Γ in a lattice in $\theta(G)$. (The conclusion in that case is still of interest, but it can be deduced in a much more elementary manner, and in fact is true for any non-compact simple Lie group. See [6], e.g.) On the other hand, for a lattice Γ with very few G-related groups, Corollary 11 can be quite strong. An example at this extreme is given by the following result.

THEOREM 12. *Let n be a prime. Then there is a cocompact lattice* $\Gamma \subset \mathrm{SL}(n,\mathbf{R})$ *such that any Lie group L intersecting* Γ *in a lattice in L must be solvable.*

This result follows immediately from:

THEOREM 13. (Kottwitz) *Let p be a prime and D be a central division algebra over* $\mathbf{Q}$ *of dimension* p^2. *Let G be the algebraic* $\mathbf{Q}$*-group isomorphic over* $\mathbf{R}$ *to* $\mathrm{SL}(p,\mathbf{R})$ *and whose* $\mathbf{Q}$*-points consist of the elements of reduced norm* 1 *in* $D^\times$. *Let H be a connected reductive* $\mathbf{Q}$*-subgroup of G, with* $H \neq G$. *Then H is a torus.*

To see that Theorem 12 follows from Theorem 13, we simply take $\Gamma = G(\mathbf{Z})$, where G is as in Theorem 13. Thus Γ is a cocompact lattice in $G(\mathbf{R}) = \mathrm{SL}(p,\mathbf{R})$. If $L \subset \mathrm{SL}(p,\mathbf{R})$ is connected and intersects Γ in a lattice, then $M =$ the algebraic hull of $L \cap \Gamma$ is a $\mathbf{Q}$-group. Applying Theorem 13 to a Levi factor we deduce that the connected component of M is solvable, and hence that L contains a solvable lattice. This implies that L itself is solvable.

PROOF OF THEOREM 13. For ease of notation we shall use H, G to denote the $\mathbf{Q}$-points of these groups. The maximal $\mathbf{Q}$-tori in $D^\times$ are of the form $E^\times$ for fields E with $\mathbf{Q} \subset E \subset D$ and $[E:\mathbf{Q}] = p$. Let $\mathcal{G} = \mathrm{Gal}(\mathbf{Q}^-/\mathbf{Q})$. The $\mathbf{Q}$-subtori of $E^\times$ correspond bijectively to the $\mathcal{G}$-invariant subspaces of the finite dimensional $\mathbf{Q}$-vector space $V = X^*(E^\times) \otimes_{\mathbf{Z}} \mathbf{Q}$. Note that V is a permutation representation of $\mathcal{G}$ with the $\mathcal{G}$-set $\mathrm{Hom}_{\mathbf{Q}\text{-alg}}(E,\mathbf{Q}^-)$ as basis. Let J denote the image of $\mathcal{G}$ in the group of permutations of the p element set $\mathrm{Hom}_{\mathbf{Q}\text{-alg}}(E,\mathbf{Q}^-)$. Then $p = [E:\mathbf{Q}]$ divides $|J|$, so J contains an element σ of order p, which must act on $\mathrm{Hom}_{\mathbf{Q}\text{-alg}}(E,\mathbf{Q}^-)$ as a p-cycle. Any $\mathcal{G}$ invariant subspace of V is invariant for the cyclic group C of order p generated by σ, and since V is isomorphic as a C-module to the group algebra $\mathbf{Q}[C]$, it follows that the only C-invariant subspaces of V correspond to the four obvious $\mathcal{G}$-invariant subspaces. Therefore, the only $\mathbf{Q}$-subtori of $E^\times$ are: $\{e\}$, $\mathbf{Q}^\times$, $E^\times$, and $T_E = \ker\{N: E^\times \to \mathbf{Q}^\times\}$, where in the latter N is the norm map. The maximal $\mathbf{Q}$-tori in G are the tori T_E; and hence their only $\mathbf{Q}$-subtori are $\{e\}$ and T_E.

Now consider a maximal $\mathbf{Q}$-torus in the connected reductive $\mathbf{Q}$-subgroup H. It is a subtorus of some T_E, and hence is either $\{e\}$ or T_E. If it is $\{e\}$, then H is also trivial. If it is T_E, then H has the same reductive rank as G, which, since H is an inner form of $\mathrm{SL}(p)$, implies that $H = G \cap D'$ for some semisimple $\mathbf{Q}$-subalgebra of D containing E. But since $\dim_E D'$ must divide $\dim_E D = p$, the subalgebra D' must be either D or E. The former case does not arise since $H \neq G$, and the latter case gives $H = T_E$.

In the context of Corollary 11, we consider the following example. Let n be prime and large, and let Γ be as in Theorem 12. Let $G = \mathrm{SL}(3,\mathbf{R})$. Let Ad be the adjoint representation of $\mathrm{SL}(n,\mathbf{R})$. Then for any non-trivial homomorphism $\theta : G \to \mathrm{SL}(n,\mathbf{R})$, we have a real analytic engaging ergodic action of G on the $n^2 - 1$ dimensional manifold $\mathrm{SL}(n,\mathbf{R})/\Gamma$. Let M be a compact manifold of dimension m. Then by Corollary 11 and Theorem 12 we deduce

$$c_m(g) \geq \min\{h(\mathrm{Ad}(\theta(g))) \mid \theta : G \to \mathrm{SL}(n,\mathbf{R}) \text{ is a non-trivial representation}\}.$$

For m small enough relative to n it is clear that this is impossible. (One can, in principle, compute both these numbers precisely.) One can of course do the same for any simple Lie group G of higher real rank. Thus:

COROLLARY 14. *Let G be a connected simple Lie group with finite center and* $\mathbf{R}$-rank$(G) \geq 2$. *Fix a positive integer m. Then there is a finitely generated group* Γ (*which we may take to be the group* Γ *in Theorem* 12) *with the properties*:

(i) *There exists a compact real analytic manifold X with an isogeny $\pi_1(X) \to \Gamma$ such that there is a real analytic, connection preserving, volume preserving, engaging, ergodic action of G on X.*

(ii) *For any compact smooth manifold M with* $\dim(M) \leq m$ *and for which there is an isogeny $\pi_1(M) \to \Gamma$, there is no smooth volume preserving action of G on M which is either engaging, or topologically engaging, or real analytic preserving a real analytic connection.*

Corollary 14 represents a new type of phenomenon. In [7],[3] conditions are given under which a group cannot appear as the fundamental group of such an M in any dimension. Here we see that some groups may appear as the fundamental group, but only in a sufficiently large dimension.

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